Discrete Mathematics 66 (1987) 231–247
North-Holland


# ON LOCAL CONVEXITY IN GRAPHS

Martin FARBER*
*AT&T Bell Laboratories, Holmdel, NJ 07733, U.S.A.*

Robert E. JAMISON**
*Mathematisches Institut, Universität Freiburg, Freiburg, Fed. Rep. Germany*

and

*Clemson University, Clemson, SC 29631, U.S.A.*

Received 26 August 1985
Revised 29 September 1986

A set $K$ of nodes of a graph $G$ is geodesically convex (respectively, monophonically convex) if $K$ contains every node on every shortest (respectively, chordless) path joining nodes in $K$. We investigate the classes of graphs which are characterized by certain local convexity conditions with respect to geodesic convexity, in particular, those graphs in which balls around nodes are convex, and those graphs in which neighborhoods of convex sets are convex. For monophonic convexity, these conditions are known to be equivalent, and hold if and only if the graph is chordal. Although these conditions are not equivalent for geodesic convexity, each defines a generalization of the class of chordal graphs. A persistent theme here will be the analogies between these graphs and chordal graphs.

## 1. Introduction

Throughout this paper $G$ will denote a connected, undirected graph without multiple edges. ($G$ may be infinite.) A *chord* of a (simple) path $x_0x_1 \dots x_n$ is an edge $x_ix_j$, where $j > i+1$. A chord of a cycle is defined similarly. A graph $G$ is *chordal* if every cycle in $G$ of length greater than 3 has a chord.

Thus far in the study of convexity in graphs (see, e.g., [1, 2, 4, 5, 7, 9–12, 14, 18, 19, 23, 24]), two types of convexity have played a prominent role. A set $K$ of nodes in $G$ is *g-convex* (respectively, *m-convex*), if, for any pair of nodes $x$, $y$ in $K$, all nodes on all shortest (respectively, chordless) paths from $x$ to $y$ also lie in $K$. Although geodesic convexity (that is, $g$-convexity) may appear the more natural of the two, it seems to be less well-behaved in general. One goal here is to obtain for geodesic convexity the analogues of some results on local convexity already derived for $m$-convexity and some related convexities in [7].

For any set $S$ of nodes in $G$ and any integer $j \geq 0$, the (closed) *neighborhood of radius j about S*, denoted $N^j[S]$, is $\{x : d_G(x, s) \leq j \text{ for some } s \text{ in } S\}$, where $d_G$ is the distance function in $G$. For brevity, we may use $d(x, y)$ instead of $d_G(x, y)$

* Research supported in part by the Natural Sciences and Engineering Research Council of Canada.
**Research supported by a Fellowship from the Alexander von Humboldt-Stiftung.

when the meaning is clear from the context. We also write $N[S]$ instead of $N^1[S]$, and $N^j[x_1, x_2, \ldots, x_n]$ if the elements of $S$ are explicitly given. For any notion of convexity on the node set of $G$, at least four degrees of local convexity may be distinguished:

$$N[v] \text{ is convex for every node } v \text{ of } G, \tag{1.1}$$

$$N^j[v] \text{ is convex for every node } v \text{ of } G \text{ and every } j \geqslant 0, \tag{1.2}$$

$$N[K] \text{ is convex for every convex subset } K \text{ of } G, \tag{1.3}$$

$$N^j[K] \text{ is convex for every convex subset } K \text{ of } G \text{ and every } j \geqslant 0. \tag{1.4}$$

Since for any set $S$ in any graph, $N^{j+1}[S] = N[N^j[S]]$, it follows that (1.3) always implies the formally stronger (1.4) for any convexity on graphs.

In [7] it was shown that conditions (1.1)–(1.4) are all equivalent for $m$-convexity and hold if and only if the graph is chordal. For $g$-convexity, conditions (1.1)–(1.3) are not equivalent. Since $g$-convexity is weaker than $m$-convexity, these conditions define generalizations of chordal graphs, and a persistent theme here will be the analogies between these graphs and chordal graphs.

Henceforth, the term "convex" will apply only to $g$-convexity unless explicitly noted to the contrary.

After an initial version of this paper was written, we learned of a paper by Soltan and Chepoi [20] which contains some overlapping results. In particular, [20] Theorem 3 is a characterization of those graphs satisfying (1.3) which is identical to that given in Theorem 3.4, parts (a) and (c), of this paper, and [20] Theorem 2 is a characterization of those graphs satisfying (1.2) which is similar to that given in Theorem 2.2 of this paper, although our characterization appears to be more natural and succinct. Also, [20] Lemma 3 is equivalent to Theorem 3.1 of this paper, although the equivalence is not entirely obvious, and, again, the statement of our result seems to be more natural and succinct. Finally, Corollary 6.7 of this paper is also a corollary of [20] Theorem 1. We have decided to include these results for several reasons. First, our development of the subject has been used extensively in a paper by the first author [6], in which a recursive characterization of the finite graphs satisfying (1.3) is presented, and, to a lesser extent, in a paper by the second author [13], in which the null-homotopy of the graphs satisfying (1.3) is studied. Second, our proofs are different from those in [20]. And, third, including these results and proofs makes this paper basically self-contained, and allows for a more cohesive presentation.

## 2. Local convexity around nodes

It is easy to check that the graphs which are locally geodesically convex in sense (1.1) are those in which every 4-cycle has a chord. The proof is left to the reader.

**Theorem 2.1.** *In a graph $G$, $N[v]$ is g-convex for every node $v$ if and only if every 4-cycle of $G$ has at least one chord.*

To investigate conditions (1.2)–(1.4), we shall need an extension of the notion of a chord of a cycle. A shortest path $B$ joining two nodes $p$ and $q$ of a cycle $C$ is a *bridge* of $C$ provided $B$ is shorter than each of the two arcs of $C$ between $p$ and $q$. (We say $p$ is *bridged* to $q$ if such a bridge exists.) $B$ is *proper* if it contains no nodes of $C$ except $p$ and $q$. Thus a chord of $C$ is a (necessarily proper) bridge of $C$ of length one. A cycle $C$ of a graph $G$ is *well-bridged* if, for each node $p$ of $C$, either the two neighbors of $p$ on $C$ are adjacent, or there is a bridge from $p$ to another node of $C$. Note that, by this definition, every 3-cycle is trivially well-bridged.

Notice that, in an $n$-cycle $C$, there is a bridge from $p$ to $q$ if and only if $d_G(p, q) < d_C(p, q)$, where $d_C$ denotes distance along the cycle $C$, i.e., in the subgraph consisting of the $n$ nodes and $n$ edges of $C$. (Thus a cycle has a bridge if and only if it is not an isometric subgraph.) An *antipode* of a node $p$ of $C$ is a node of $C$ at maximum distance from $p$ along $C$. Thus $p$ has one antipode when $n$ is even and two antipodes when $n$ is odd. By the triangle inequality, one can easily check that there is a bridge at $p$ if and only if $d_G(p, v) < d_C(p, v)$ for at least one antipode $v$ of $p$ in $C$. Thus $C$ is well-bridged if and only if, for each $p$ in $C$, either the neighbors of $p$ in $C$ are adjacent, or $d_G(p, v) < d_C(p, v)$ for some antipode $v$ of $p$ in $C$.

**Theorem 2.2.** *In any graph $G$, $N^j[v]$ is g-convex for all nodes $v$ and every $j > 0$ if and only if every cycle in $G$ of length other than 5 is well-bridged.*

**Proof.** Suppose first that $N^j[v]$ is convex for all $v$ and $j$. Then we have the following immediate consequence:

**(2.3)** For any node $p$ if $G$, if $p$ has neighbors $x \neq y$ which are of distance at most $j$ from some node $v$, then either $xy$ is an edge, or $d_G(p, v) \leq j$.

Now let $p$ be a node on an $n$-cycle $C$, with $n \neq 3, 5$. If $n$ is even, then $p$ has a unique antipode $v$, and the desired conclusion follows at once from (2.3).

Thus suppose $n$ is odd, with $n = 2j + 3$, $j > 1$. Let $C$ be $px_1x_2 \ldots x_jvw y_jy_{j-1} \ldots y_1p$, so $v$ and $w$ are the antipodes of $p$ on $C$. If $d(x_j, y_j) \geq 3$, then $x_jwvy_j$ is a shortest $x_j - y_j$ path, so $v, w \in N^j[p]$ by convexity, and we have the desired bridge at $p$. Thus we may assume $d(x_j, y_j) \leq 2$.

Assuming $d_G(p, v) = d_G(p, w) = j + 1$, we shall show that $x_1$ and $y_1$ must be adjacent. First, we show that $d(y_1, x_j) \leq j$.

By assumption $px_1 \ldots x_jv$ is a shortest $p$–$v$ path, so $d(p, x_j) = j$. Since $j > 1$, $y_2$ exists and

$$d(y_2, x_j) \leqslant d(y_2, y_j) + d(y_j, x_j) \leqslant j - 2 + 2 = j,$$

so by (2.3) either $p$ and $y_2$ are adjacent, or $d(y_1, x_j) \leqslant j$. But $py_1y_2 \ldots y_jw$ is a shortest $p$–$w$ path, so $p$ and $y_2$ are not adjacent; hence, $d(y_1, x_j) \leqslant j$, as desired.

Now $d(y_1, w) = j$ and $d(y_1, x_j) \leqslant j$. Thus, by (2.3), either $x_j$ and $w$ are adjacent, or $d(y_1, v) \leqslant j$. In the first case, $px_1 \ldots x_jw$ and $py_1 \ldots y_jw$ are both shortest paths. Thus $d(w, x_1) = d(w, y_1) = j$, and $d(w, p) = j + 1$. Hence, $x_1$ and $y_1$ are adjacent, by (2.3). In the second case, $d(y_1, v) \leqslant j$, $d(x_1, v) = j$ and $d(p, v) = j + 1$. Thus we may conclude from (2.3) that $x_1y_1$ is an edge, as desired.

To prove the converse, suppose every cycle in $G$ of length other than 5 is well-bridged. We will show, by contradiction, that $N^j[v]$ is $g$-convex for every node $v$ and every positive integer $j$. Suppose $N^j[v]$ is not convex for some $v$ and $j$. Among all such pairs $(v, j)$, choose one which minimizes $j$. (Notice that $j \geqslant 2$ by Theorem 2.1.) Consider all pairs of vertices $\{a, b\} \subseteq N^j[v]$ such that $I(a, b)$ is not contained in $N^j[v]$, where

$$I(a, b) = \{u : u \text{ lies on a shortest } a\text{–}b \text{ path}\}.$$

Among these, choose a pair $\{a, b\}$ which minimizes the distance from $a$ to $b$ in $N^j[v]$ and, subject to this, minimizes $d_G(a, b)$. Let $P = au_1u_2 \ldots u_mb$ be a shortest $a$–$b$ path which meets $G \backslash N^j[v]$. We distinguish two cases:

(i) $m = 1$. Then $u_1$ is not in $N^j[v]$, so $d(v, a) = d(v, b) = j$. Let $vP_1a$ and $bP_2v$ be shortest $v$–$a$ and $b$–$v$ paths, respectively. Observe that the minimality of $j$ assures that $P_1 \cap P_2 = \{v\}$. Thus $C = vP_1au_1bP_2v$ is an even cycle in which $v$ and $u_1$ are antipodes. Since $ab$ is not an edge, $u_1$ is bridged to $v$, i.e., $u_1 \in N^j[v]$, which is a contradiction.

(ii) $m > 1$. Let $Q = ay_1y_2 \ldots y_kb$ be a shortest $a$–$b$ path in $N^j[v]$. Then $m \leqslant k$. Also, $I(a, y_k) \cup I(y_1, b) \subseteq N^j[v]$, by the choice of $\{a, b\}$. It follows that $P \cap Q = \{a, b\}$. Thus $C' = au_1u_2 \ldots u_mby_ky_{k-1} \ldots y_1a$ is a cycle of length at least 6 in which neither $a$ nor $b$ lies on a bridge. Consequently $u_1y_1$ and $u_my_k$ are edges. Since $I(y_1, b) \subseteq N^j[v]$, we find that $y_1u_1 \ldots u_mb$ is not a shortest $y_1$–$b$ path. Thus, $m \geqslant k$, and so $m = k$.

If $u_1 \in N^j[v]$, then $I(u_1, b) \nsubseteq N^j[v]$, $d_{N^j[v]}(u_1, b) \leqslant d_{N^j[v]}(a, b)$ and $d_G(u_1, b) < d_G(a, b)$, contradicting the choice of $a$ and $b$. Thus, $u_1 \notin N^j[v]$. Similarly $u_m \notin N^j[v]$. Moreover, $u_1u_2 \ldots u_my_my_{m-1} \ldots y_1u_1$ is an even cycle in which $u_1$ and $y_m$ are antipodes. Thus, either $y_1$ and $u_2$ are adjacent, or $u_1$ is bridged to $y_m$. However, in either case, we deduce that $I(y_1, b) \cup I(a, y_m) \nsubseteq N^j[v]$, which is a contradiction. □

## 3. Local convexity about convex sets

A graph $G$ is *bridged* if every cycle of length greater than 3 has a bridge. Equivalently, $G$ is bridged if it contains no isometric cycles (other than triangles).

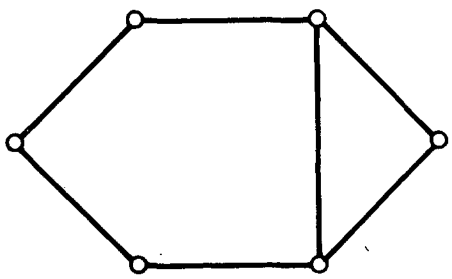

Fig. 1

For geodesic convexity, the difference between local convexity conditions (1.2) and (1.3) is just a matter of chords in 5-cycles, as will be shown in Theorem 3.4. First, however, we prove an analogue of a basic property of chordal graphs. Namely, we show that all cycles are well-bridged provided all (nontrival) cycles are bridged. (If 5-cycles are omitted, this does not hold, as shown by the graph in Fig. 1.)

**Theorem 3.1.** *If $G$ is a bridged graph, then every cycle in $G$ is well-bridged.*

This theorem is an immediate consequence of the following proposition.

**Proposition 3.2.** *Suppose that $G$ is a graph and $C$ is a minimum length cycle in $G$ which is not well-bridged. Then $C$ has no bridges.*

**Proof.** Suppose, on the contrary, that $C$ has a bridge. It is convenient to show first that $C$ has no chords. Indeed, any chord $ab$ of $C$ splits $C$ into smaller cycles, one of which, say $C'$, contains a node $p$ at which $C$ is not well-bridged. Clearly, $p \neq a, b$, so the neighbors of $p$ in $C'$ are the same as its neighbors in $C$. Hence these are not adjacent since $C$ is not well-bridged at $p$. By the choice of $C$, it follows that $p$ is bridged in $C'$ to some node $v$ of $C'$. Since the chord $ab$ introduces no new nodes, $v$ is also in $C$ and we have $d_G(p, v) < d_{C'}(p, v) \leq d_C(p, v)$, so $p$ is bridged to $v$ in $C$, contradicting the choice of $p$. Hence, $C$ has no chords.

Now let $n = 2k + \varepsilon$, where $\varepsilon = 1, 2$, and let $C$ be $v_0v_1v_2 \ldots v_{2k+\varepsilon}$, where $v_{2k+\varepsilon} = v_0$. We claim that there is an $i$ such that $C$ is not bridged at $v_i$ but is bridged at $v_{i+k}$ (addition modulo $n = 2k + \varepsilon$). Suppose not, and let $v_0$ be a node at which $C$ is not bridged. Then, by supposition, $C$ is not bridged at any node in the sequence $v_0, v_k, v_{2k}, v_{3k}, \ldots$. If $n$ is odd, then this sequence includes all nodes of $C$. Whence $C$ has no bridge, contrary to hypothesis. If $n$ is even, then $v_i$ and $v_{i+k+1}$ are antipodes and hence either both bridged or both not bridged. By supposition, if $C$ is not bridged at $v_i$, it is not bridged at $v_{i+k}$ and hence not bridged at $v_{i+2k+1} = v_{i-1}$. Thus $C$ is not bridged at any node of the sequence $v_0, v_{2k+1}, v_{2k}, v_{2k-1}, \ldots,$ again contrary to hypothesis. Thus the claim is established.

Thus, without loss of generality, we may assume $C$ is not bridged at $v_0$ but is bridged at $v_k$. Let $m \geqslant 0$ be the least index so that $v_k$ is bridged to $v_m$. Since $v_0$ is not bridged to $v_k$ and $C$ has no chords, it follows that $k+3 \leqslant m < 2k+\varepsilon$. Let $v_k = u_0u_1 \ldots u_{t-1}u_t = v_m$ be a shortest path from $v_k$ to $v_m$. Let $r$ be the largest index such that $u_r \in \{v_0, v_1, \ldots, v_k\}$. Then $v_k$ is not bridged to $u_r$, so the minor arc of $C$ from $v_k$ to $u_r$ is a shortest path and we may assume $u_i = v_{k-i}$ for $0 \leqslant i \leqslant r$.

Let $s$ be the least index such that $u_s \in \{v_{k+1}, v_{k+2}, \ldots, v_{2k+\varepsilon-1}\}$. Since $v_0$ is not bridged to $v_{k+1}$, it follows that $d_G(v_m, u_s) = d_C(v_m, u_s)$. Since $d_G(v_k, u_r) = d_C(v_k, u_r)$, it follows from the triangle inequality that $v_k$ and $u_r$ are both bridged to $u_s$. Thus $u_s \in \{v_m, v_{m+1}, \ldots, v_{2k+\varepsilon-1}\}$, by choice of $m$. Moreover, as above, we may assume $u_{t-i} = v_{m+i}$ for $0 \leqslant i \leqslant t-s$.

Now, by our choice of $r$ and $s$, the cycle

$$C': v_0v_1 \ldots v_{k-r} = u_ru_{r+1} \ldots u_s = v_{m+t-s} \ldots v_{2k+\varepsilon} = v_0$$

is a proper cycle. Since $u_r$ is bridged to $u_s$, it follows that $C'$ is shorter than $C$ and that $v_0 \neq u_r$; whence $v_1$ and $v_{2k+\varepsilon-1}$ are both in $C'$. Now $C'$ is well-bridged by minimal choice of $C$. Since the neighbors $v_1, v_{2k+\varepsilon-1}$ of $v_0$ in $C'$ are not adjacent, $v_0$ must be bridged in $C'$. Since $v_0$ is not bridged in $C$, it follows that $v_0$ is bridged in $C'$ to some $u_j$ with $r<j<s$.

For even $n$, we need some additional information. Consider the cycle $C'': v_k = u_0u_1 \ldots u_t = v_mv_{m-1} \ldots v_k$. This is a proper cycle by our choice of $m$ and our normalization of $u_su_{s+1} \ldots u_t$. Since $u_0 \ldots u_t$ is a bridge of $C$, $C''$ is shorter than $C$ and hence is well-bridged. Since $u_0u_1 \ldots u_t$ is a shortest $u_0 - u_t$ path, $v_k = u_0$ is not bridged in $C''$ to any $u_i$. By choice of $m$, $v_k$ is not bridged to any of $v_{k+1}, v_{k+2}, \ldots, v_{m-1}$ in $C$, and hence in $C''$. Thus $v_k$ is not bridged in $C''$, so its neighbors $u_1$ and $v_{k+1}$ must be adjacent. Thus $v_{k+1}u_1u_2 \ldots u_t$ is a path, so $d_G(v_{k+1}, u_i) \leqslant i = d_G(v_k, u_i)$ for all $i>0$.

Now set $w = v_k$ if $n$ is odd, and set $w = v_{k+1}$ if $n$ is even. Then we have

$$\begin{aligned} d_G(v_0, w) &\leqslant d_G(v_0, u_j) + d_G(u_j, w) < d_{C'}(v_0, u_j) + d_G(u_j, w) \\ &\leqslant d_G(v_0, u_s) + d_G(u_s, u_j) + d_G(u_j, v_k) \\ &= d_G(v_0, u_s) + d_G(u_s, v_k) < d_G(v_0, u_s) + d_C(u_s, v_k) \\ &= k+\varepsilon. \end{aligned}$$

Since all distances are integral, each strict inequality yields a difference of at least 1. Thus $d_G(v_0, w) \leqslant k+\varepsilon-2$. But $d_C(v_0, w) = k+\varepsilon-1$. Hence $v_0$ is bridged to $w$ after all, yielding the desired contradiction. $\square$

**Lemma 3.3.** *Let $K$ be a g-convex set in a bridged graph $G$. Then $d_{K\cup y}(x, y) = d_G(x, y)$, for each $x$ in $K$ and $y$ in $N[K]\backslash K$.*

**Proof.** Assume the assertion fails. Among all pairs $(x, y)$ with $x$ in $K$, $y$ in $N[K]\backslash K$ and $d_{K\cup y}(x, y) > d_G(x, y)$, choose one which minimizes $d_G(x, y)$. Note that $d_G(x, y) \geqslant 2$.

Let $xu_1u_2 \ldots u_sy$ be a $G$-shortest $x$–$y$ path. If $u_i \in K$ for some $i$, then $u_1, u_2, \ldots, u_i \in K$, since $K$ is convex, and $d_G(u_i, y) = d_{K \cup y}(u_i, y)$, since $d_G(u_i, y) < d_G(x, y)$. Hence $d_G(x, y) = d_{K \cup y}(x, y)$, contrary to the choice of $x$ and $y$. Thus $u_i \notin K$ for all $i$.

Let $xv_1v_2 \ldots v_ty$ be a $K \cup y$-shortest $x$–$y$ path. Then $xv_1 \ldots v_t$ is a $G$-shortest $x$–$v_t$ path, since $K$ is convex. By assumption, $t > s \geq 1$. Moreover, $s + 2 > t$, since $xu_1u_2 \ldots u_syv_t$ is an $x$–$v_t$ path of length $s + 2$ which meets $G \backslash K$, and $K$ is convex. Thus $t = s + 1$.

Now, $d_G(v_1, y) \leq d_{K \cup y}(v_1, y) = t = d_G(x, y)$. Thus, $d_G(v_1, y) = d_{K \cup y}(v_1, y)$, by the choice of $(x, y)$. Hence, $y$ is not bridged to any node of the cycle $xu_1u_2 \ldots u_syv_{s+1}v_s \ldots v_1x$. Thus $u_s$ and $v_{s+1}$ are adjacent, by Theorem 3.1, and so $xu_1u_2 \ldots u_sv_{s+1}$ is a $G$-shortest $x$–$v_{s+1}$ path which meets $G \backslash K$, contradicting the fact that $K$ is convex. □

**Theorem 3.4.** *For any graph $G$, the following are equivalent*:

(a) *$N^j[K]$ is g-convex for any g-convex set $K$ and any $j > 0$;*

(b) *$N^j[v]$ is g-convex for any node $v$ and any $j > 0$, and every 5-cycle in $G$ has a chord.*

(c) *$G$ is a bridged graph.*

**Proof.** To check that (a) implies (b), we need only check that any 5-cycle $C = abcde$ has a chord. Indeed, as an edge, $\{b, c\}$ is trivially convex. Thus, $N[b, c]$ is convex by (a). Now $a, d \in N[b, c]$. Thus if $ad$ is not a chord of $C$, then $dea$ is a shortest path, so $e \in N[b, c]$, by convexity. But this says either $be$ or $ce$ is an edge.

By Theorem 2.2, (b) implies (c).

To show (c) implies (a), it suffices to prove that $N[K]$ is convex for any convex $K$. Assume, on the contrary, that $K$ is convex, but $N = N[K]$ is not convex. Then, for some $x$, $y$ in $N$, there is a shortest path $xu_1u_2 \ldots u_sy$ with $u_i \notin N$ for some $i$. Among all such pairs $\{x, y\}$, select one which minimizes $d_G(x, y)$. From this minimal choice, it follows that $u_i \notin N$ for all $i$. Let $xv_1v_2 \ldots v_ty$ be a $K \cup \{x, y\}$-shortest $x$–$y$ path. By Lemma 3.3, it follows that $xv_1v_2 \ldots v_t$ is a $G$-shortest path, since it is shortest with respect to $K \cup \{x\}$. Thus $x$ lies on no bridge of the cycle $C = xv_1 \ldots v_tyu_s \ldots u_1x$. By Theorem 3.1, $C$ is well-bridged, so $v_1u_1$ must be an edge. Hence $u_1 \in N$, which is a contradiction. □

Recall that a bridge of a cycle $C$ is *proper* if it meets $C$ only in its endpoints. In a sense, the proper bridges are the correct analogues of chords, since they permit the cycle $C$ to be split into two smaller cycles. Using the triangle inequality, it is easy to see that if a cycle has a bridge, then it also has a proper bridge. Refining this idea, we shall now prove an extension of Theorem 3.1. A cycle $C$ is *properly well-bridged* if, for each $p$ in $C$, either the two neighbors of $p$ on $C$ are adjacent, or there is a proper bridge from $p$ to another node of $C$.

**Theorem 3.5.** *Suppose $G$ is a bridged graph. Then every cycle in $G$ is properly well-bridged.*

**Proof.** The proof is by induction on the length of the cycle. Clearly any cycle of length at most 5 in $G$ is properly well-bridged. Let $C = v_0v_1 \ldots v_nv_0$ be a cycle in $G$, and suppose that every cycle which is shorter than $C$ is properly well-bridged. Suppose $v_1$ and $v_n$ are not adjacent. By symmetry, it suffices to show that $v_0$ lies on a proper bridge of $G$. Since $G$ is bridged, $v_0$ lies on some bridge of $C$, by Theorem 3.1. For each bridge $B = u_0u_1u_2 \ldots u_r$ between $v_0 = u_0$ and another node of $C$, let $m(B) = \max\{k : u_i = v_i \text{ for } i = 0, 1, \ldots, k, \text{ or } u_i = v_{-i} \text{ for } i = 0, 1, \ldots, k\}$ (addition modulo $n+1$). Choose such a bridge, $B = v_0u_1 \ldots u_r$, which maximizes $m(B)$ and, subject to this condition, minimizes $r$. Let $s = \min\{t : t > m(B) \text{ and } u_t \in C\}$. Then $s = r$, for otherwise $v_0$ is not bridged to $u_s$, and we could replace $v_0u_1 \ldots u_s$ by the minor arc from $v_0$ to $u_s$ in $C$ to obtain a bridge $B'$ between $v_0$ and $u_r$ with $m(B') > m(B)$. If $m(B) = 0$, then $B$ is a proper bridge. Suppose $m(B) > 0$. Then we may assume that $u_1 = v_1$.

Let $j$ satisfy $u_r = v_j$. Then $m(B) + 2 \leqslant j \leqslant n - m(B) - 1$. Also, $C' = v_0v_1 \ldots v_{m(B)}u_{m(B)+1} \ldots u_rv_{j+1} \ldots v_nv_0$ is a cycle which is shorter than $C$. Since $v_1$ and $v_n$ are not adjacent, $v_0$ lies on a proper bridge of $C'$, by the choice of $C$. Since $d_G(v_0, u_i) = i$, for $i = 1, 2, \ldots, r$, $v_0$ must be properly bridged to some $v_k$ with $j < k < n$. Let $B^* = v_0y_1y_2 \ldots y_tv_k$ be such a bridge of $C'$. Then $B^*$ is a bridge of $C$. Moreover, either $B^*$ is a proper bridge of $C$, or there is an $i$ such that $y_i = v_l$, for some $m(B) < l < j$. In the latter case, choose the least such $i$. Since $d_G(v_0, y_i) = i$, we have $i \leqslant d_C(v_0, y_i)$. If $i = d_C(v_0, y_i)$, let $P$ be obtained from $B^*$ by replacing $v_0y_1 \ldots y_i$ by the minor arc of $C$ from $v_0$ to $y_i$. Then $P$ is a bridge of $C$ with $m(P) > m(B)$, contradicting the choice of $B$. Thus $i < d_C(v_0, y_i)$, and so $v_0$ is properly bridged to $y_i$ in $C$. □

The preceding result does not generalize to the graphs in which all cycles of

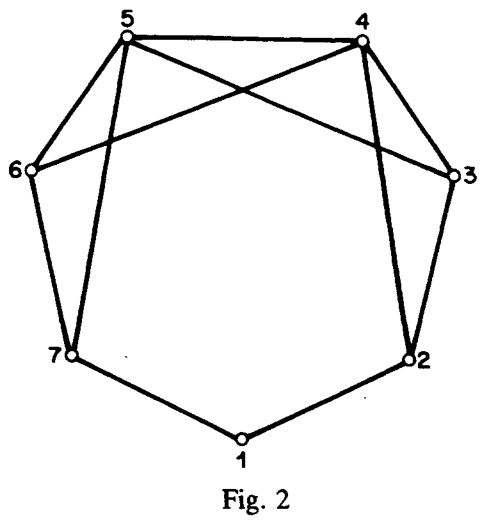


Fig. 2

length other than 5 are well-bridged. Indeed, the graph in Fig. 2 has diameter 2, and so, in any $n$-cycle, $n \geq 6$, every node is bridged to its antipode(s). Since all 4-cycles have chords, it follows that all cycles of length other than 5 in this graph are well-bridged. But, the 7-cycle shown fails to have a proper bridge at vertex 1.

## 4. Recognizing bridged graphs

The local convexity results of the last section permit the development of a polynomial time recognition algorithm for bridged graphs. This is based on the following observation.

**Theorem 4.1.** *Let $G$ be a finite connected graph on $n$ nodes, and let $S$ be a set of nodes of $G$. Then one can determine in $O(n^3)$ steps whether $S$ is g-convex.*

**Proof.** As is well-known, the distance matrix $d_G$ of $G$ may be computed in $O(n^3)$ steps [17]. Now $y$ is on a shortest path from $x$ to $z$ if and only if

$$d_G(x, z) = d_G(x, y) + d_G(y, z). \tag{4.2}$$

It thus suffices to test whether (4.2) holds for all triples $(x, y, z)$, with $x, z \in S$ and $y \in G \setminus S$. □

**Theorem 4.3.** *Let $G$ be a finite connected graph on $n$ nodes. Then one can determine in $O(n^4)$ steps whether $G$ is bridged.*

**Proof.** By Theorem 3.4, checking that $G$ is bridged is equivalent to checking that (a) $N^j[v]$ is convex for all $v$ and $j$, and (b) all 5-cycles have chords. For an arbitrary graph, verifying condition (b) seems to require $O(n^5)$ steps. However, as we will see, this can be reduced to $O(n^4)$ steps if condition (a) holds. We first show how to check if condition (a) holds in $O(n^4)$ steps.

First compute the distance matrix $d_G$ of $G$. Now for each vertex $v$, $G$ can be partitioned into the shells $S^j[v] = \{x : d_G(v, x) = j\}$ in linear time. If $N^j[v]$ is not convex, there is a shortest path between two nodes of $S^j$ which passes outside of $N^j[v]$ and hence through a node of $S^{j+1}$. Thus to check the convexity of $N^j[v]$, it suffices to check if (4.2) holds for each triple $(x, y, z)$ with $x, z \in S^j$ and $y \in S^{j+1}$. In this way, a given triple $(x, y, z)$ is associated with at most one shell. Hence only $O(n^3)$ steps are required for any $v$ to check whether $N^j[v]$ is convex for all $j$. Thus $O(n^4)$ steps are required in all.

If $N^j[v]$ is not convex for some $j$ and $v$, then $G$ is not bridged. Otherwise, we check for chordless 5-cycles, as follows.

Consider each triple $u$, $v$, $w$ of vertices. If $vw$ is an edge and $u$ is at distance 2 from both $v$ and $w$, then check to see if there is a common neighbor $x$ of $v$ and $u$ which is not adjacent to $w$, and a common neighbor $y$ of $w$ and $u$ which is not adjacent to $v$. If so, then $uxvwyu$ is a chordless 5-cycle, since there are no chordless 4-cycles. Given the distance matrix, it is easy to see that the required number of steps per triple is $O(n)$. Thus $O(n^4)$ steps are required in all. □

The number of triples required to be checked to test the convexity of $N^j[v]$ may be further reduced via an observation involving a weakened version of geodesic convexity. For any fixed integer $k>0$, a set $K$ of nodes of $G$ is *$g_k$-convex* if, for any $x$, $y$ in $K$ with $d_G(x, y) \leq k$, all shortest paths from $x$ to $y$ lie in $K$. Condition (2.3) used in the proof of Theorem 2.2 asserts in essence that $N^j[v]$ is $g_2$-convex. In the proof of the direct implication of Theorem 2.2, the convexity of $N^j[v]$ is used mostly in the weaker form (2.3), with one exception, where paths of length 3 are needed. Thus we have the following result.

**Theorem 4.4.** *For any graph $G$, every cycle in $G$ of length other than 5 is well-bridged if and only if $N^j[v]$ is $g_3$-convex for each node $v$ and positive integer $j$.*

Note that if $G$ is an odd cycle, then $N^j[v]$ is $g_2$-convex for all $v$ and $j$. Thus $g_2$-convexity of point neighborhoods is insufficient to insure the existence of bridges. However, the following result is available in connection with Theorem 3.4.

**Theorem 4.5.** *A graph $G$ is bridged if and only if*
(a) *$N^j[v]$ is $g_2$-convex for each node $v$ and positive integer $j$, and*
(b) *$N^j[v, w]$ is $g_2$-convex for each edge $vw$ and positive integer $j$.*

**Proof.** That the above local convexity conditions are necessary for $G$ to be bridged follows from Theorem 3.4(a). To see that they are sufficient, let $C$ be an $n$-cycle, $n>3$.

Let $p$ be a node of $C$. If $n$ is even, let $v$ be the unique antipode of $p$ and write $n=2j+2$. Since $N^j[v]$ is $g_2$-convex, either $p$ is bridged to $v$, or the neighbors of $p$ on $C$ are adjacent. If $n$ is odd, then the antipodes $v$ and $w$ of $p$ are adjacent. Let $n=2j+3$. Since $N^j[v, w]$ is $g_2$-convex, either $p \in N^j[v, w]$ (and hence $p$ is bridged), or its neighbors are adjacent. □

## 5. Constructions and examples

Let $S$ be a subset of the nodes of a graph $G$. An *$S$-piece* of $G$ is a set of the form $A \cup S$ where $A$ is a component of $G \setminus S$.

**Lemma 5.1.** *If $S$ is g-convex in a graph $G$, then every $S$-piece of $G$ is also g-convex in $G$.*

**Proof.** If $P$ is a path joining two nodes of an $S$-piece which exits the $S$-piece, then it must exit through a node $p$ of $S$ and reenter through a node $q$ of $S$. If $P$ is a shortest path, then by the convexity of $S$, $P$ cannot exit $S$ between $p$ and $q$. □

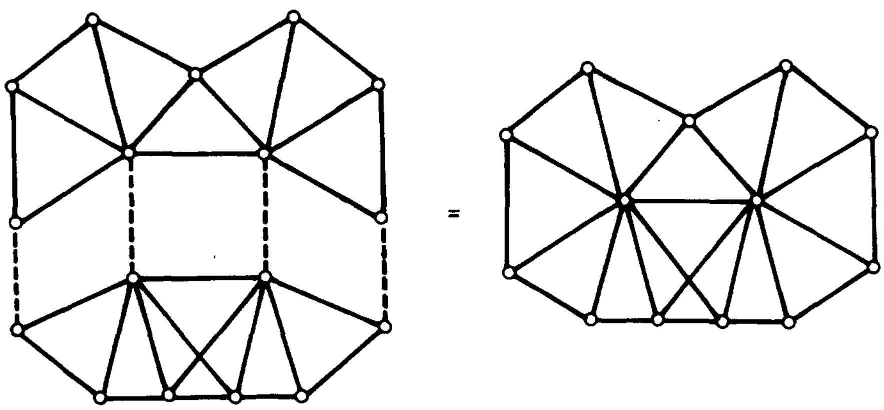

Fig. 3

**Theorem 5.2.** *If $S$ is $g$-convex in a graph $G$, then $G$ is bridged if and only if every $S$-piece of $G$ is bridged.*

**Proof.** By Lemma 5.1, the $S$-pieces of $G$ are convex. Thus, if $G$ is bridged, so are the $S$-pieces.

Conversely, suppose each $S$-piece is bridged. Let $C$ be a cycle of length $n > 3$ in $G$. If $C$ lies in a single $S$-piece, then it has a bridge, by supposition. Thus assume $C$ passes through two different components, $A$ and $B$, of $G \backslash S$. There are then nodes $p$ and $q$ of $S$ on $C$ such that one arc $C_1$ of $C$ from $p$ to $q$ passes through $A$, and the other arc $C_2$ passes through $B$. Since $S$ is convex, neither $C_1$ nor $C_2$ can be a shortest path. Thus the distance from $p$ to $q$ in $G$ is less than that along $C$, so $p$ and $q$ are bridged in $C$. □

We shall say that $G$ is a *$g$-sum* of subgraphs $H_1$ and $H_2$ provided

$$G = H_1 \cup H_2, \tag{5.3}$$

$$S = H_1 \cap H_2 \text{ is } g\text{-convex}, \tag{5.4}$$

and

$$H_1 \text{ and } H_2 \text{ are unions of } S\text{-pieces of } G. \tag{5.5}$$

Thus, by Theorem 5.2, any $g$-sum of bridged graphs is bridged. Figure 3 shows a $g$-sum of two chordal graphs.

The analogue of the above development for chordal graphs and $m$-convexity is valid and easily established. As is well-known [3], in any chordal graph, a minimal set separating any pair of nonadjacent nodes is complete, and hence trivially $m$-convex. Thus any incomplete chordal graph is the $m$-sum of smaller chordal graphs, and the class of chordal graphs is the smallest class closed under $m$-sums and containing all complete graphs.

It is natural to wonder if all bridged graphs can be built up by $g$-sums from chordal graphs (and hence from complete graphs). Unfortunately, this is not the case, as will be shown in Theorem 5.8. Nevertheless, it is possible to generate a very diverse class of bridged graphs by $g$-sums of chordal graphs.

**Example 5.6.** Let $T(4, 4)$ be the graph whose nodes and edges are those of the regular square tessalation of the plane. We shall form a new graph $G$ from $T(4, 4)$ by inserting one diagonal into each square of $T(4, 4)$. For each vertical strip of squares, select an orientation $+$ or $-$. If $+$ is selected for a strip, insert the diagonal with slope $+1$ into each square of that strip; otherwise, insert the diagonal with slope $-1$. Each strip thus becomes an (infinite) chordal graph (in fact, a unit interval graph).

Moreover, since the diagonals of all squares in a vertical strip have the same orientation, any finite subpath of a vertical line is the unique shortest path joining its end nodes. Thus each vertical line is $g$-convex. Therefore, $G$ is a $g$-sum of chordal graphs, and hence is bridged. Notice that $G$ contains chordless cycles of all lengths except 4, 5, and 7.

If the same orientation is selected for all strips, then the result of the above construction is an affine deformation of the vertex-edge graph $T(6, 3)$ of the regular triangular tesselation. In fact, any choice of orientations yields a graph isomorphic to $T(6, 3)$. But the method may be altered to produce a variety of graphs.

The construction can be modified by first taking a closed Jordan curve $J$ in $T(4, 4)$, and then concentrating on the subgraph $J^*$ of nodes and edges lying on or inside $J$. (This need not be an induced subgraph, as two nodes on $J$ may be joined by an edge outside $J$.) Each vertical strip now breaks into components, and we are free to choose the orientations $+$ and $-$ independently for each component. The resulting graph $G$ can be built by a sequence of $g$-sums from the (chordal) components of the vertical strips. Thus $G$ is bridged.

Bridged graphs lack an important hereditary property which chordal graphs enjoy. Namely, an induced subgraph of a bridged graph need not be bridged. Clearly any bridge of a 4-cycle or 5-cycle must be a chord. Since chords are hereditary, in any induced subgraph $H$ of a bridged graph $G$, all 4-cycles and 5-cycles must have chords. By the construction in (5.7), this characterizes the possible induced subgraphs of bridged graphs. This fact, along with Theorem 3.4, emphasizes the importance of 5-cycles in bridged graphs, and suggests, along with Theorem 4.5 and 6.1, that a special role is played by bridged graphs of diameter 2, as follows. If $G$ is bridged, then $N[v]$ induces a bridged graph of diameter 2 for each vertex $v$, by Theorem 3.4. The sets $N[v]$, $v \in V(G)$, form a cover of $G$. Theorem 6.1 says that a set is convex if and only if its intersection with each set in this cover is convex. Thus the structure of convex sets in bridged graphs is determined by piecing together convex sets in bridged graphs of diameter 2. The class of bridged graphs of diameter 2 (not all of which arise by the method of Example 5.7) consists of all graphs without induced 4-cycles or 5-cycles in which every maximal clique is a dominating set [15].

**Example 5.7.** Let $H$ be a graph in which every 4-cycle and every 5-cycle has a chord. Form $G$ by adjoining to $H$ a new vertex $u$ adjacent to every node of $H$. Every $n$-cycle $(n > 3)$ through $u$ thus has a chord. Any $n$-cycle $(n > 3)$ not through $u$ either has a bridge through $u$, if $n > 5$, or a chord, by choice of $H$, if $n = 4, 5$. Thus $G$ is a bridged graph of diameter 2.

Notice that the $g$-convex subsets of $G$ are of two kinds: (1) complete subgraphs of $H$, and (2) sets of the form $K \cup \{u\}$ where $K$ is $g_2$-convex in $H$.

A special case of this construction is the wheel $W_n$ formed by taking $H$ to be an $n$-cycle. This is bridged if $n > 5$. Any three consecutive nodes on the rim together with the axle $u$ form a $g$-convex subset. By Theorem 5.2 we may glue together along such subsets to produce larger bridged graphs. This allows the construction of bridged graphs with many chordless 7-cycles.

**Theorem 5.8.** *There exists an incomplete bridged graph which is not a g-sum of smaller graphs.*

**Proof.** Let $H$ be the graph shown in Fig. 4. $H$ is bipartite and hence contains no 5-cycles. That $H$ contains no 4-cycles is also easy to see. Thus $H$ vacuously satisfies the requirements of Example 5.7. Let $G$ be the bridged graph formed from $H$ by the adjunction of a single universal node $u$ as in Example 5.7. We will show that no $g$-convex subset of $G$ separates $G$ into two or more components, so $G$ cannot be a $g$-sum of smaller graphs.

Suppose, on the contrary, that $S$ is a convex set that separates two points $p$ and $q$ of $H$. Let $R = G \backslash S$, and let $P$ and $Q$ denote the components of $R$ containing $p$ and $q$, respectively. For all choices of $p$ and $q$, we shall examine how the other points of $H$ must be distributed between $S$ and $R$ and show that a contradiction always results. We begin by showing that no node $p$ of the outer eight cycle can be separated from any other node $q$ of $H$. By rotational symmetry, we may assume $p = 1$:

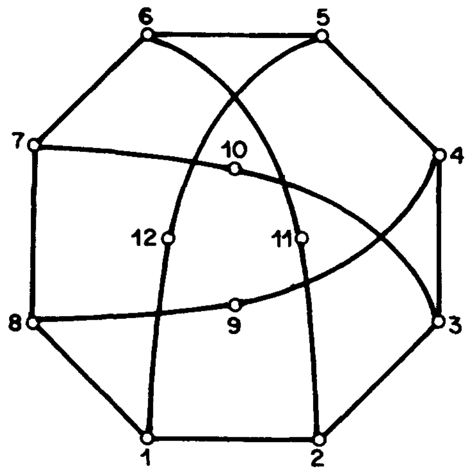


Fig. 4

$q = 5$. Then $12 \in S$. Since $1 \notin S$, a convex set, and 12-1-2 is a path, it follows that $2 \in R$. Likewise, $8, 4, 6 \in R$. As 2 is adjacent to 1, and 4 is adjacent to 5, we have $2 \in P$ and $4 \in Q$. Thus to separate 2 and 4, we must have $3 \in S$. Likewise, $11 \in S$ to separate 2 and 6. But 3-2-11 is a path, so $2 \in S$ by convexity, contradicting $2 \in R$.

$q = 3$. Then $2 \in S$, which forces 4 and 8 to be in $R$. Thus $9 \in S$, which forces $5 \in R$. But then 3-4-5 is in $R$, so $5 \in Q$, and hence 5 is separated by $S$ from 1, contrary to the above case.

$q = 4$. Then, if either 3 or 5 were in $R$, they would be in $Q$, and hence separated from 1, since they are adjacent to 4. This would contradict the preceding cases, so $3, 5 \in S$. But then $4 \in S$, by convexity, a contradiction.

The cases $q = 7$ and $q = 6$ now follow by symmetry.

$q = 10$. If either 3 or 7 were in $R$, they would be separated from 1, contrary to the preceding. But $3, 7 \in S$ implies $10 \in S$, a contradiction.

$q = 11$. Then $2 \in S$. Moreover, $6 \in S$, since otherwise it is separated from 1. But this implies the contradiction $11 \in S$.

The case $q = 9$ now follows by symmetry.

This completes the proof that no node on the outer eight cycle can be separated from any other node. But since $R$ has at least two components $P$ and $Q$, any node of $R$ is separated from some other node. Thus $S$ contains the outer eight cycle. But then, by convexity, $S$ must be all of $H$, an absurdity. □

In spite of this negative result, the class of finite bridged graphs does have a recursive characterization [6]. Also, it is shown in [13] that every cycle in a bridged graph is formed by 'zipping' together chordal graphs.

## 6. Other local convexity properties

The next result below is an analogue of a classical theorem of Tietze [21, 22] which states that in Euclidean space, any locally convex continuum is convex. An analogue of this result was already shown in [7] to hold for $m$-convexity in chordal graphs.

**Theorem 6.1.** *If $K$ is a connected set of nodes in a bridged graph $G$ and $N[v] \cap K$ is g-convex for each $v$ in $K$, then $K$ is g-convex.*

**Proof.** Supposing $K$ is not convex, we shall find a $v \in K$ such that $N[v] \cap K$ is not convex. Since $K$ is not convex, there exists a pair of nodes, $\{a, b\}$, in $K$ such that $I(a, b) \nsubseteq K$ (where $I(a, b)$ is as defined in the proof of Theorem 2.2.). Among all such pairs, choose a pair $\{a, b\}$ which minimizes $d_K(a, b)$ and, subject to this, minimizes $d_G(a, b)$. (Since $K$ is connected, $d_K(a, b) < \infty$.) Let $P = au_1u_2 \ldots u_mb$ be a shortest $a$–$b$ path meeting $V \setminus K$, and let $Q = ay_1y_2 \ldots y_kb$ be a $K$-shortest $a$–$b$ path. Then, by the choice of $a$ and $b$, we have $I(a, y_k) \cup I(y_1, b) \subseteq K$. Note that, just as in the proof of sufficiency of Theorem 2.2, the assumption that $m > 1$ leads to a contradiction. (Recall that in that particular part of the proof we did not use the assumption that the set we were dealing with was of the form $N^j[v]$.)

Thus $m = 1$. Now, $C = au_1by_ky_{k-1} \ldots y_1a$ is a cycle in which $a$ lies on no bridge. Thus $u_1$ and $y_1$ are adjacent. Since $I(y_1, b) \subseteq K$ and $u_1 \notin K$, we conclude that $y_1$ and $b$ are adjacent. Thus $N[y_1] \cap K$ is not convex. $\square$

We note that this theorem cannot be extended to include all graphs satisfying (1.2), e.g., let $G$ be a 5-cycle and let $K$ consist of any four nodes. Then $N[v] \cap K$ is convex for each $v \in K$. On the other hand, letting $v^*$ be the node not in $K$, we find that $N[v^*] \cap K$ is not convex. The next result shows that this situation holds for all graphs satisfying (1.2).

**Theorem 6.2.** *If $K$ is a connected set of nodes in a graph satisfying* (1.2) *and $N[v] \cap K$ is g-convex* (*and hence complete*) *for each node $v$ not in $K$, then $K$ is g-convex.*

**Proof.** The proof follows that of Theorem 6.1, except that once we conclude $m = 1$, then we are done, since $N[u_1] \cap K$ is not convex. $\square$

Since any $g$-convex set is trivially $g_2$-convex, another way to state the above theorem is:

**(6.3)** In a graph in which every cycle of length other than 5 is well-bridged, a connected set of nodes is $g$-convex if and only if it is $g_2$-convex.

The next result states that bridged graphs are locally connected.

**Theorem 6.4.** *If $v$ is a non-cutvertex of a bridged graph $G$, then $N(v) = N[v] \backslash \{v\}$ is connected.*

**Proof.** Note that since $v$ is not a cutvertex, any two edges at $v$ lie in a common block of $G$ and hence in some cycle.

Now assume the result fails. Among all pairs of neighbors of $u$ lying in different components of $N(v)$, choose a pair, $x$, $y$, minimizing $d_{G \backslash v}(x, y)$. Clearly, $x$ and $y$ are not adjacent. Let $P = xu_1u_2 \ldots u_ky$ be a $G \backslash v$ shortest $x$–$y$ path. Observe that $v$ is not adjacent to $u_1$, by the choice of $x$ and $y$. Thus $x$ lies on a proper bridge of the cycle $vxu_1u_2 \ldots u_kyv$, by Theorem 3.5, contradicting the choice of $P$. $\square$

We note that this result does not hold for a 5-cycle, and hence does not hold for all graphs satisfying (1.2).

We close with a result of very general character. By a *convexity structure* on a set $X$, we mean a collection $L$ of subsets of $X$ which is closed under intersection, and contains both the empty set and $X$. The members of $L$ are called *convex sets*. The smallest member of $L$ containing a set $S \subseteq X$ is the *hull* of $S$, and is denoted

$L(S)$ (cf. [12]). Suppose $X$ is endowed with a metric, $d(\cdot, \cdot)$. For each $x \in X$ and each $r \in \mathbb{R}^+ \cup \{\infty\}$, the *ball of radius* $r$ about $x$, denoted $B^r(x)$, is $\{y \in X : d(x, y) \leq r\}$.

Let $S$ be a subset of $X$. Then the diameter of $S$, denoted diam($S$), is $\sup\{d(x, y) : x, y \in S\}$, the *radius* of $S$ with respect to $X$ is $\inf\{r : S \subseteq B^r(x)$ for some $x \in X\}$, and the *center* of $S$ with respect to $X$ is $\{x \in X : S \subseteq B^r(x)\}$, where $r$ is the radius of $S$. Observe that, in general, the center of a nonempty set may be empty. However, one can show by standard arguments that if $X$ is compact, then the center of every nonempty subset of $X$ is nonempty. In particular, this is true if $X$ is finite. We note that the standard definition of the center of a graph [8] coincides with that given here by letting $X = S$ be the entire node set of the graph, and letting $d(\cdot, \cdot)$ be the standard distance function of the graph.

**Theorem 6.5.** *Suppose* $(X, d)$ *is a metric space and* $L$ *is a convexity structure on* $X$ *in which all balls are convex. Then*:

(i) *The center of any convex set is convex*;

(ii) *For each* $S \subseteq X$, $\text{diam}(L(S)) = \text{diam}(S)$.

**Proof.** The validity of (i) follows from the observation that the center of $S$ is $\bigcap \{B^r(x) : x \in S\}$, where $r$ is the radius of $S$.

To prove (ii), let $j = \text{diam}(S)$, and set

$$T = \bigcap \{B^j[v] : v \in S\} \quad \text{and} \quad U = \bigcap \{B^j[v] : v \in T\}.$$

Clearly $S \subseteq T$, since $j$ is the diameter of $S$. Thus $U \subseteq T$, since $U$ is an intersection over a larger set. Moreover, $S \subseteq U$, by definition of $T$. Now, for any $v$, $w$ in $U$, we have $v \in T$, and hence $w \in B^j[v]$, by definition of $U$. Hence $\text{diam}(U) \leq j$. By assumption, each $B^j[v]$ is convex, so $U$ is convex. Thus $L(S) \subseteq U$, so $\text{diam}(L(S)) \leq j$. Since $S \subseteq L(S)$, the reverse inequality follows. □

**Corollary 6.6.** *The center of a connected graph satisfying* (1.2) *is connected. In particular, the center of any connected bridged graph is connected.*

**Proof.** Convex sets in connected graphs induce connected subgraphs. □

We note that essentially the same proof was used in [12] to show that the center of any connected chordal graph is connected (cf. [16]).

Now, for a graph $G$, and set $S$ of nodes of $G$, let conv($S$) be the smallest $g$-convex set containing $S$, and let

$$\text{diam}_G(S) = \sup\{d_G(x, y) : x, y \in S\}.$$

**Corollary 6.7.** *For any subset* $S$ *of nodes of a graph* $G$ *satisfying* (1.2),

$$\text{diam}_G(\text{conv}(S)) = \text{diam}_G(S).$$

*In particular, this equality holds if* $G$ *is bridged.*

## References


[1] L.M. Batten, Geodesic Subgraphs, J. Graph Theory (1983) 159–163.
[2] J.M. Delire, Graphs with high Radon number, Bull. Acad. Roy. Sci. Belgique, to appear.
[3] G.A. Dirac, On rigid circuit graphs, Abh. Math. Seminar Univ. Hamburg 25 (1961) 71–76.
[4] P. Duchet, Convex sets in graphs II: minimal path convexity, J. Combin. Theory Ser. B, to appear.
[5] P. Duchet and H. Meyniel, Ensemble convexes dans les graphes I, European J. Combin. 4 (1983) 127–132.
[6] M. Farber, Bridged graphs and geodesic convexity, Discrete Math., this issue.
[7] M. Farber and R.E. Jamison, Convexity in graphs and hypergraphs, SIAM J. Alg. Disc. Meth. 7 (1986) 433–444.
[8] F. Harary, Graph Theory (Addison–Wesley, Reading, MA, 1969).
[9] F. Harary and J. Nieminen, Convexity in graphs, J. Differential Geom. 16 (1981) 185–190.
[10] S.P.R. Hebbare, A class of distance convex simple graphs, Ars Combin. 7 (1979) 19–26.
[11] R.E. Jamison, Partition numbers for trees and ordered sets, Pac. J. Math. 96 (1981) 115–140.
[12] R.E. Jamison, A perspective on abstract convexity: classifying alignments by varieties, in: D.C. Kay and M. Breen, eds., Convexity and Related Combinatorial Geometry (Marcel Dekker, New York, 1982).
[13] R.E. Jamison, On the null-homotopy of bridged graphs, Europ. J. of Combin., to appear.
[14] R.E. Jamison and R. Nowakowski, A Helly theorem for convexity in graphs, Discrete Math. 51 (1984) 35–39.
[15] R.E. Jamison and J. Pach, private communication (1984).
[16] R. Laskar and D. Shier, On powers and centers of chordal graphs, Discrete Appl. Math. 6 (1983) 139–147.
[17] E.L. Lawler, Combinatorial Optimization: Networks and Matroids (Holt, Rinehart and Winston, New York, 1976).
[18] H.M. Mulder, The interval function of a graph, Math. Centre Tracts 132, Amsterdam (1980).
[19] V.P. Soltan, $d$-convexity in graphs, Soviet Math. Dokl. 28 (1983) 419–421.
[20] V.P. Soltan and V.D. Chepoi, Conditions for invariance of set diameters under $d$-convexification in a graph, Cybernetics 19 (1983) 750–756.
[21] H. Tietze, Über Konvexheit im Kleinen, Math. Z. 28 (1928) 697–707.
[22] F.A. Valentine, Convex Sets (McGraw Hill, San Francisco, 1964).
[23] P. Vanden Cruyce, A convexity problem in 3-polytopal graphs, Arch. Math. 43 (1984) 84–88.
[24] P. Vanden Cruyce, A convex characterization of the graphs of the dodecahedron and icosahedron, Discrete Math. 50 (1984) 99–105.